Recall: Curvature is

$$\mathrm{K} = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}'(t)\times\vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^{3}}$$

Entry Task: $f(x) = x^2 - 6x$
At what point (x, y, z) is the curvature maximum?

2D Curvature:

For, $y = f(x)$, we form:

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle.$$

$$\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$$

$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

13.4 Position, Velocity, Acceleration

If ***t = time*** and position is given by

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$

then

$$\boldsymbol{r}'(t) = \lim_{h\to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$
$$= \frac{\text{change in position}}{\text{change in time}}$$
$$= \text{velocity} = \boldsymbol{v}(t)$$

$$|\boldsymbol{r}'(\boldsymbol{t})| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed} = \frac{\text{ds}}{\text{dt}}$$

$$\boldsymbol{r}''(t) = \lim_{h\to 0} \frac{\boldsymbol{r}'(t+h) - \boldsymbol{r}'(t)}{h}$$
$$= \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \text{acceleration} = \boldsymbol{a}(t)$$

Example: Let t be **time in seconds** and assume the position of an object (in **feet**) is given by $\boldsymbol{r}(t) = < t, 2e^{-t}, 0 >$ Compute $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$, and $\boldsymbol{r}''(t)$ and $\boldsymbol{r}'(0)$, $|\boldsymbol{r}'(0)|$, and $\boldsymbol{r}''(0)$.

HUGE application:
Modeling ANY motion problem.

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \text{ , so}$$

$$\boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant)

If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve a differential equation) to find velocity and position.

That is how we can model ALL motion problems!

HW Example: An object of mass 10 kg is being acted on by the force

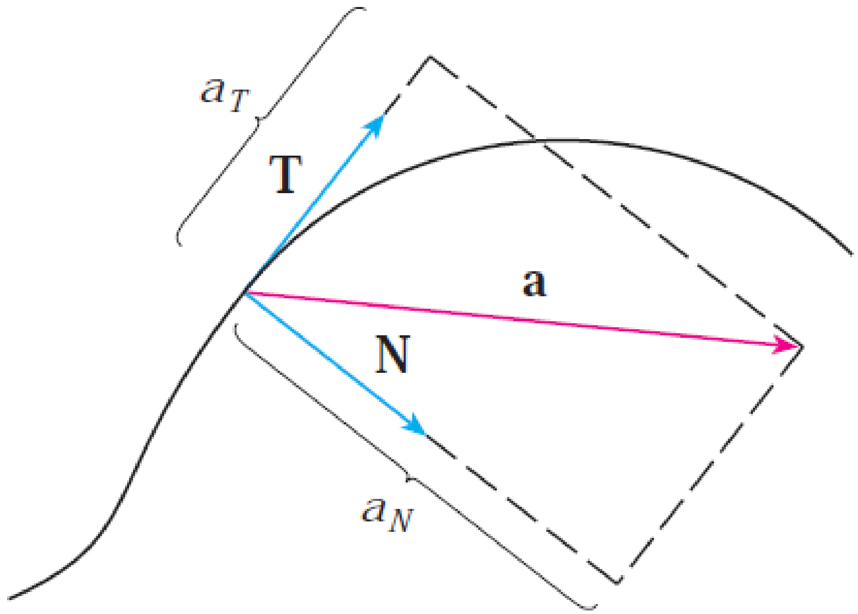
$$\boldsymbol{F} = \langle 130t, 10e^{t}, 10e^{-t} \rangle.$$

You are given

$$\boldsymbol{v}(0) = \langle 0, 0,1 \rangle \text{ and } \boldsymbol{r}(0) = \langle 0, 1,1 \rangle.$$

Find the position function.

Measuring and describing acceleration



Recall: $\mathrm{comp}_{\boldsymbol{b}}(\boldsymbol{a}) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b}}$ = length.

We define the tangential and normal components of acceleration by:

$a_T = \mathrm{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{T}$ = tangential

$a_N = \mathrm{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{N}$ = normal

For computing use,

$$a_T = \frac{\vec{\boldsymbol{r}}' \cdot \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} \quad \text{and} \quad a_T = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|}$$

For interpreting use,

$a_T = v' = \frac{d}{dt}|r'(t)| =$ “deriv. of speed”

$a_N = kv^2 =$ curvature · (speed)^2

Example:

$$\vec{\boldsymbol{r}}(t) = < \cos(t)\,, \sin(t)\,, t >$$

Find the tangential and normal components of acceleration.

Deriving interpretations:

Note that: $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$

Let $v(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

1. $\vec{\boldsymbol{T}}(t) = \dfrac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \dfrac{\vec{\boldsymbol{v}}(t)}{v(t)}$ implies $\vec{\boldsymbol{v}} = v\vec{\boldsymbol{T}}$.

2. $\kappa(t) = \dfrac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \dfrac{|\vec{\boldsymbol{T}}'|}{v(t)}$ implies $|\vec{\boldsymbol{T}}'| = \kappa v$.

3. $\vec{\boldsymbol{N}}(t) = \dfrac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \dfrac{\vec{\boldsymbol{T}}'}{\kappa v}$ implies $\vec{\boldsymbol{T}}' = \kappa v\vec{\boldsymbol{N}}$.

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = v'\vec{\boldsymbol{T}} + v\vec{\boldsymbol{T}}', \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = v'\vec{\boldsymbol{T}} + kv^2\vec{\boldsymbol{N}}.$$

Conclusion:

$a_T = v' = \dfrac{d}{dt}|r'(t)| =$ "deriv. of speed"

$a_N = kv^2 =$ curvature · (speed)^2